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Homological conditions for graphical splittings of antisocial graph groups

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Abstract

We define an *antisocial graph group* to be a graph group arising from a graph whose clique graph is triangle free. In every dimension, the existence of graphical splittings of an antisocial graph group G is shown to correspond to nonvanishing of Betti numbers of the ball of radius 1 in the well-known cube complex on which G acts freely. This can be regarded as a first step towards generalising Stallings' ends theorem to higher dimensions.

MSC: 20F32; 05C25

1. Introduction

In [6] (see also [7]), Stallings proved the well-known theorem that for a finitely generated group G, $e(G) \geqslant 2$ (i.e., is equal to 2 or is infinite) if and only if G splits over a finite subgroup, either as a free product with amalgamation or as an HNN extension. An elegant proof has been given by Dunwoody (see [3]) using the Bass–Serre Theory of groups acting on trees. A good account is also given in [8]. The motivation for this paper is to find analogues of Stallings' ends theorem in higher dimensions. The number of ends $e(G)$ is defined by looking at the number of infinite connected components of the complement of the ball of radius n in a Cayley graph of G, and taking the limit as n tends to infinity. To generalise this, we could consider higher homology groups of the complement of the ball of radius n in a suitable generalisation of the Cayley graph, take the limit of their ranks as n tends to infinity, and try to relate these to the existence of "n-dimensional" splittings.

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The matter is somewhat simplified in this paper. Firstly, we focus our attention on *graph groups*, which have an interesting range of branching behaviour, if considered as geometric objects. Secondly, information can be obtained from only considering the homology of the boundary of the ball of radius 1 in their generalised Cayley graphs. (We call the boundary of this ball a "sphere" even though it is, in general, a union of conventional topological spheres.) The main result shows that, even in this restricted case, the correspondence between nonvanishing of higher dimensional Betti numbers and splittings is considerably more complicated than Stallings' ends theorem, although a basic dichotomy between free products with amalgamation and HNN extensions still exists.

2. Graph groups

Let X be a finite graph with vertex set V and edge set E (elements of E are subsets $\{v_1, v_2\}$ of V of cardinality 2). The *graph group* $G(X)$ defined by X is the group presented by the generating set V and the set $\{[v_1, v_2] \mid \{v_1, v_2\} \in E\}$ of relators. (Graph groups are sometimes called *right-angled Artin groups*.) Let $\mathcal{X}$ be the category of graphs and graph morphisms and let $\mathcal{G}$ be the category of groups and homomorphisms. (Specifically, since graph-theoretic terminology varies widely, I mean the following: If $X = (V, E)$ is a graph, where elements of E are subsets $\{v_1, v_2\}$ of V of cardinality 2, then a *graph morphism* is a map from V to V such that for all $\{v_1, v_2\} \subset V$, $\{v_1, v_2\} \in E$ if and only $\{f(v_1), f(v_2)\} \in E$.) If $f : X \to Y$ is a morphism in $\mathcal{X}$ then by Von Dyck's theorem there is an induced morphism $G(f) : G(X) \to G(Y)$ in $\mathcal{G}$ defined on generators v of $G(X)$ by $G(f)(v) = f(v)$. Moreover $G : \mathcal{X} \to \mathcal{G}$ is a functor.

Example 2.1. (1) If X is the empty graph then $G(X)$ is the trivial group.

(2) If $X = K_n$, the complete graph of order n, then $G(X) \cong \mathbb{Z}^n$.

(3) If $X = \overline{K_n}$, the totally disconnected graph of order n, then $G(X) \cong F_n$, the free group of rank n.

(4) Let $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ be graphs. Then the $\mathcal{X}$*-join* $V_1 + V_2$ of V_1 and V_2 is defined to be the graph

$$\bigl(V_1 \cup V_2, E_1 \cup E_2 \cup \bigl\{\{v_1, v_2\} \mid v_1 \in V_1,\ v_2 \in V_2\bigr\}\bigr).$$

Clearly $G(X_1) \times G(X_2) \cong G(X_1 + X_2)$. Thus the class of graph groups is closed under direct products and includes, for example, the group $F_m \times F_n$, which arises from the complete bipartite graph K_{mn}. See Fig. 1.

Special cases of the $\mathcal{X}$-join are the $\mathcal{X}$*-cone* $\mathcal{C}(X) = X + \overline{K_1}$ and the $\mathcal{X}$*-suspension* $\mathcal{S}(X) = X + \overline{K_2}$ of X (see Fig. 2). These correspond to the groups $G(X) \times \mathbb{Z}$ and $G(X) \times F_2$, respectively. Thus, for example, $F_2 \times F_2$ arises from the square graph $\mathcal{S}(\overline{K_2})$ and the n-fold product $F_2 \times \cdots \times F_2$ arises from the graph $\mathcal{S}^n(\overline{K_2})$.

(5) Let X_1, X_2 and Y be graphs and let $i_1 : Y \to X_1$ and $i_2 : Y \to X_2$ be injective $\mathcal{X}$-morphisms. We denote their $\mathcal{X}$-pushout by $X_1 \cup_Y^{i_1,i_2} X_2$, usually suppressing i_1 and i_2

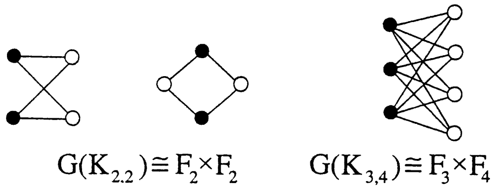


Fig. 1.

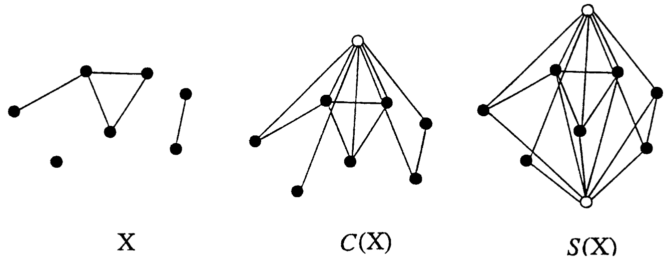


Fig. 2.

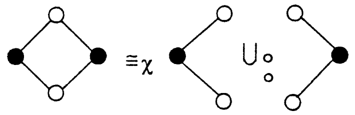


Fig. 3.

from the notation. It is clear from their presentations that $G(X_1 \cup_Y X_2)$ is isomorphic to the free product with amalgamation $G(X_1) *_{G(Y)}^{G(i_1),G(i_2)} G(X_2)$. In other words, the G-functor respects pushouts. For example, if P_2 denotes the path graph of length 2 then $\mathcal{S}(\overline{K_2}) = P_2 \cup_{\overline{K_2}} P_2$. Thus (see Fig. 3)

$$F_2 \times F_2 \cong (F_2 \times \mathbb{Z}) *_{F_2} (F_2 \times \mathbb{Z}).$$

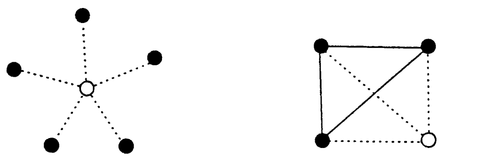
We shall be interested in such pushouts where neither i_1 nor i_2 are isomorphisms. We call these *proper* pushouts; they give rise to nontrivial free products with amalgamation.

(6) If $i : Y \to X$ is an injective $\mathcal{X}$-morphism then we write $\mathcal{C}_Y^i(X)$ for the $\mathcal{X}$-pushout $X \cup_Y^i \mathcal{C}(Y)$. It is clear from their presentations that $G(\mathcal{C}_Y^i(X))$ is isomorphic to the HNN extension $G(X) *_{G(Y)}^{G(i)}$. For example, the group arising from the star graph $K_{1,n}$ with $n+1$ vertices is isomorphic to $F_n *_{F_n}$. Note also that, for all $n \geqslant 1$, since $K_n = \mathcal{C}(K_{n-1})$ we have (see Fig. 4)

$$\mathbb{Z}^n \cong \mathbb{Z}^{n-1} *_{\mathbb{Z}^{n-1}}.$$

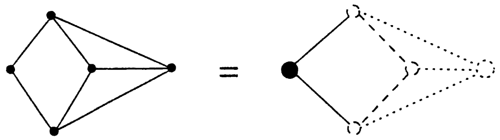
Note that an HNN extension arising in this manner is a pushout of the form

$$G\big(\mathcal{C}_Y(X)\big) \cong G(X) *_{G(Y)} G\big(\mathcal{C}(Y)\big) \cong G(X) *_{G(Y)} \big(G(Y) \times \mathbb{Z}\big).$$



$F_5 *_{F_5} \cong F_5 \times \mathbb{Z}$ $\mathbb{Z}_4 \cong \mathbb{Z}_3 *_{\mathbb{Z}_3} \cong \mathbb{Z}_3 \times \mathbb{Z}$

Fig. 4.



$(F_2 \times F_2) *_{F_2 \times \mathbb{Z}}$

Fig. 5.

In particular, if the associated subgroup $G(Y)$ is equal to $G(X)$ then the above HNN extension is isomorphic to the product $G(X) \times \mathbb{Z}$. (This is termed an *ascending* HNN extension.) This is the case in the two examples so far given. The group corresponding to the graph in Fig. 5 is an example of a non-ascending HNN extension.

We assign a cube complex Cube(X) to a graph X as follows. Let cube(X) be the combinatorial cube complex with one vertex $c_0(X)$, a labelled edge $(c_0(X), c_0(X), v)$ for each $v \in V(X)$ and a cube with edges $v_1, \ldots, v_m$ if and only if $\{v_1, \ldots, v_m\}$ generate a free Abelian subgroup of $G(X)$. Since the class of free Abelian groups is closed under taking subgroups, the faces of cubes in cube(X) are also in cube(X) and we have a complex. Define Cube(X) to be the (combinatorial) universal cover of cube(X) and write $\overline{\mathrm{Cube}(X)}$ for its geometric realisation. The 0-skeleton of Cube(X) is then $G(X)$ and the 1-skeleton is the Cayley graph of $G(X)$ with respect to V and we label it accordingly. There is a square between four elements g, h_1, h_2 and k of $G(X)$ if and only if for two distinct vertices v_1 and v_2 of X we have $h_1 = gv_1$, $h_2 = gv_2$ and $k = h_2v_1 = h_1v_2$. Similarly, there is a cube between eight elements g, h_1, h_2, h_3, k_1, k_2, k_3 and m if and only if for three distinct vertices v_1, v_2 and v_3 of X we have $h_1 = gv_1$, $h_2 = gv_2$, $h_3 = gv_3$, $k_1 = h_2v_3 = h_3v_2$, $k_2 = h_1v_3 = h_3v_1$, $k_3 = h_2v_1 = h_1v_2$ and $m = k_1v_1 = k_2v_2 = k_3v_3$. There are analogous rules for the higher dimensional cubes. See Fig. 6.

If X has no triangles then $\overline{\mathrm{Cube}(X)}$ is just the Cayley complex of $G(X)$ with respect to its graphical presentation. Cube(X) is a finite dimensional cube complex on which $G(X)$ acts freely. It is shown in [1] that the geometric realisation $\overline{\mathrm{Cube}(X)}$ is CAT(0) and (hence) contractible, so cube(X) is a $K(G(X), 1)$-space. Hence all graph groups have finite cohomological dimension and are thus torsion free. Furthermore, the cohomological dimension of $G(X)$ over $\mathbb{Z}$ is given by $\dim(\mathrm{Cube}(X)) = \dim(\mathrm{cube}(X))$, which is in

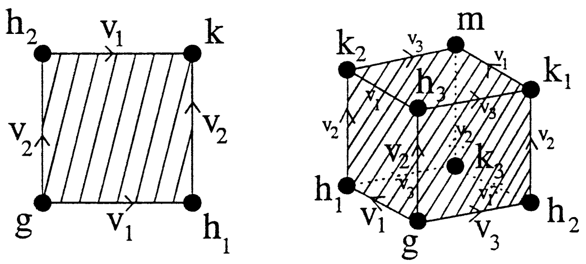


Fig. 6.

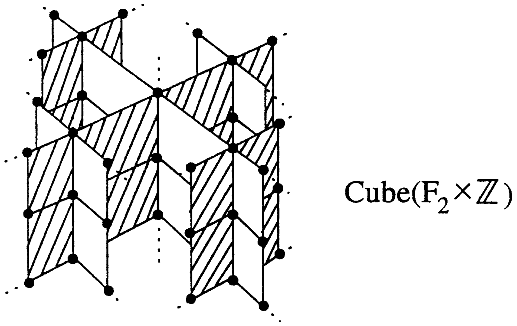


Fig. 7.

turn the largest number of vertices in a maximal complete subgraph of X. In particular, $\dim(\mathrm{Cube}(X)) \leqslant |V_X|$ with equality if and only if X is complete, i.e., $G(X)$ is free Abelian.

Example 2.2. The complex $\mathrm{Cube}(K_n)$ is homeomorphic to $\mathbb{R}^n$ whereas $\mathrm{Cube}(\overline{K_n})$ is a tree, the Cayley graph of F_n. The geometric realisation of the cube complex $\mathrm{Cube}(F_2 \times \mathbb{Z})$ is the Cayley complex of $F_2 \times \mathbb{Z}$, which is homeomorphic to the product of a tree with $\mathbb{R}$. See Fig. 7.

Graph groups are in fact biautomatic [2,9] (as are the more general *graph products* of biautomatic groups [5]). They are an important source of groups with prescribed finiteness properties, as shown in the work of Bestvina and Brady [1].

3. Clique graphs and the *X*-sphere

In this section we develop a notion of the X-*sphere* S^X where X is a finite graph. This is a union of topological spheres, where (a) the dimensions of the spheres involved, and (b) which spheres intersect, are governed by the graph X. If X is a complete graph on $m \geqslant 1$ vertices then S^X is homeomorphic to the usual sphere S^{m-1}.

First we define a *multi-labelled graph* Y to be the data

$$(V_Y, E_Y, L_Y, l_Y)$$

where (V_Y, E_Y) is a graph, L_Y is a set called the *set of labels* of Y and l_Y is a map from V_Y to 2^{L_Y} called the *label map* of Y (so that vertices of Y are labelled by subsets of L). Recall that a *clique* Q of a graph X is a maximal complete subgraph of X. We say that Q is an *n-clique* if it has n vertices.

Definition 3.1. Let X be a graph. Then the *labelled clique graph* of X, denoted $\mathcal{Q}X$, is the multi-labelled graph with $V_{\mathcal{Q}X}$ equal to the set of cliques in X and such that

(1) the edge set $E_{\mathcal{Q}X}$ is equal to the set of unordered pairs $\{Q_1, Q_2\}$ of cliques in X such that $Q_1 \neq Q_2$ and $Q_1 \cap Q_2 \neq \emptyset$;
(2) the set of labels $L'_{\mathcal{Q}X}$ is equal to V_X; and
(3) for all cliques Q of X, $l_{\mathcal{Q}X} : V_{\mathcal{Q}X} \to 2^{V_X}$ is given by $l_{\mathcal{Q}X}(Q) = V_Q$.

For convenience we label the edges of $\mathcal{Q}X$ by defining the function $l^E_{\mathcal{Q}X} : E_{\mathcal{Q}X} \to 2^{V_X}$ as

$$l^E_{\mathcal{Q}X}\bigl(\{Q_1, Q_2\}\bigr) = Q_1 \cap Q_2.$$

Hereafter we denote both $l_{\mathcal{Q}X}$ and $l^E_{\mathcal{Q}X}$ by l. Let $v \in V_{\mathcal{Q}X}$ and $e = \{v_1, v_2\} \in E_{\mathcal{Q}X}$. We define $|v|$ to be $|l(v)|$ and $|e|$ to be $|l(e)|$. Note that, by maximality of cliques, we have $|e| < |v_1|$ and $|e| < |v_2|$.

Example 3.2. (1) For a complete graph, $\mathcal{Q}K_n$ is a single point, labelled by the set of vertices of K_n.

(2) For a totally disconnected graph, $\mathcal{Q}(\overline{K_n})$ is the graph $\overline{K_n}$ with each vertex labelled by itself.

(3) A graph cannot be recovered from its unlabelled clique graph, as, for instance, all complete graphs have the same unlabelled clique graph, namely a single vertex. Labelling vertices of the clique graph removes this problem. For example, consider $X_1 = K_3 \cup_{K_1} K_3$ and let $X_2 = K_3 \cup_{K_2} K_3$, i.e., X_1 is two triangles joined at a vertex and X_2 is two triangles joined at an edge. Again, $\mathcal{Q}X_1$ and $\mathcal{Q}X_2$ both have the same underlying vertex-labelled graph, but their (distinct) labelled clique graphs are as follows. The example illustrated in Fig. 8 shows that it is not sufficient only to know the number of elements in the label of each vertex of the clique graph. We also need information about the edge labels.

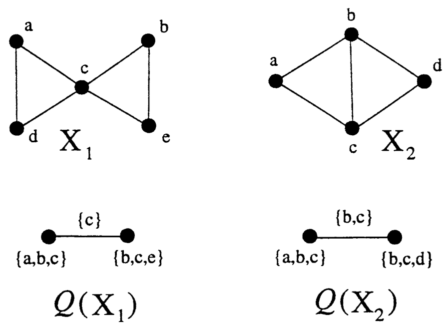


Fig. 8.

A discussion of (unlabelled) clique graphs is given in [4].

We can recover a graph from its labelled clique graph as follows. Let $K(S)$ denote the complete graph on a set S of vertices. For all $v \in V_{\mathcal{Q}X}$ let K_v be equal to the graph $K(l(v))$ and let $\sim$ be the equivalence relation on $\coprod_{v \in V_{\mathcal{Q}X}} K_v$ which identifies $K(l(v_1))$ with $K(l(v_2))$ for all $e = \{v_1, v_2\} \in E_{\mathcal{Q}X}$. We recover X as the quotient $(\coprod_{v \in V_{\mathcal{Q}X}} K_v)/\sim$. If $Y = \mathcal{Q}X$ then we write $X = \mathcal{Q}^{-1}Y$.

We consider a sphere S^n to be embedded in the usual way in $\mathbb{R}^{n+1}$. Suppose that $a_1, \ldots, a_{n+1}$ are the coordinates in $\mathbb{R}^{n+1}$. If $A \subset \{a_1, \ldots, a_{n+1}\}$ then we call the intersection of S^n with the A-hyperplane the *A-equator* of S^n. For the purposes of the next definition we define S^{-1} to be the empty topological space. (This is consistent with the usual terminology since it is the sphere of radius 1 in $\mathbb{R}^0$.)

Definition 3.3. Let X be a finite graph. For each vertex $v \in V_{\mathcal{Q}X}$ let S_v be an $m(v)$-sphere, where $m(v) = |l(v)| - 1$, and let $\sim$ be the equivalence relation on the set $s(X) = \coprod_{v \in V_{\mathcal{Q}X}} S_v$ which identifies the $l(e)$-equators of the spheres S_{v_1} and S_{v_2} whenever $e = \{v_1, v_2\} \in E(\mathcal{Q}X)$. We call the quotient space $s(X)/\sim$ the *X-sphere S^X*.

Example 3.4.

(1) If ε is the empty graph then S^ε is the empty topological space.

(2) For a complete graph, $S^{K_n} = S^{n-1}$, whereas for a totally disconnected graph $S^{\overline{K_n}}$ is the discrete topological space of cardinality $2n$.

(3) If we use the same notation as example 3.2 then $S^{X_1} \cong S^2 \cup_{S^0} S^2$ and $S^{X_2} \cong S^2 \cup_{S^1} S^2$. See Fig. 9.

Let $B_m(X)$ denote the *cubical ball* of radius m in the geometric realisation of $\mathrm{Cube}(X)$ (i.e., $B_1(X)$ is the union of all cubes which meet $1_{G(X)}$ and we recursively define $B_m(X)$ to be the union of all cubes which meet $B_{m-1}(X)$). The *sphere* $S_m(X)$ of radius m in $\mathrm{Cube}(X)$ is defined as

$$S_m(X) = \partial B_m(X) = B_m(X) - \mathrm{Int}\big(B_m(X)\big).$$

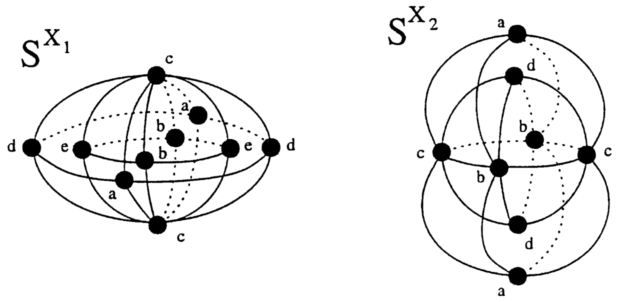


Fig. 9.

The following proposition justifies the definition of S^X.

Proposition 3.5. *Let X be a finite graph. Then $S_1(X)$ is homeomorphic to S^X.*

Proof. The ball $B_1(X)$ of radius 1 in $\mathrm{Cube}(X)$ consists of cubes attached to the ball of radius 1 in the Cayley graph $\Gamma_{V_X}G(X)$ as follows. For all subsets $A = \{v_{m_1}, \ldots, v_{m_n}\}$ of V_X there are 2^n n-cubes attached to $v_{m_1}^{\pm 1}, \ldots, v_{m_n}^{\pm 1}$ if and only if A induces a complete subgraph of X. If A does not induce a clique then $v_{m_1}^{\pm 1}, \ldots, v_{m_n}^{\pm 1}$ give rise to cubes which are not maximal and hence are in $\mathrm{Int}(B_1(X))$. Thus the only cubes in $S_1(X) = \partial B_1(X)$ are faces of cubes which arise from cliques and, in particular, those which do not meet $1_{G(X)}$. Suppose that $v_{m_1}, \ldots, v_{m_{n_1}}$ do induce a clique in X. Let $C_{m_1,\ldots,m_n}$ be the cube subcomplex of $B_1(X)$ induced by the 2^n cubes corresponding to $v_{m_1}^{\pm 1}, \ldots, v_{m_n}^{\pm 1}$. Then $\partial C_{m_1,\ldots,m_n} \cong \partial D^n \cong \partial S^{n-1}$. Suppose $v_{p_1}, \ldots, v_{p_{n_2}}$ also induce a clique in X. In $B_1(X)$ the cube subcomplexes $C_{m_1,\ldots,m_{n_1}}$ and $C_{p_1,\ldots,p_{n_2}}$ intersect in the cube subcomplex $C_{a_1,\ldots,a_{n_3}}$, where $\{a_1, \ldots, a_{n_3}\} = \{m_1, \ldots, m_{n_1}\} \cap \{p_1, \ldots, p_{n_2}\}$. Now $\partial C_{a_1,\ldots,a_{n_3}} \cong S^{n_3-1}$ so in $S_1(X)$,

$$\partial C_{m_1,\ldots,m_{n_1}} \cap \partial C_{p_1,\ldots,p_{n_2}} = \partial C_{a_1,\ldots,a_{n_3}} \cong S^{n_3-1}. \qquad \square$$

4. Antisocial graphs

Let X be a finite graph. By the *clique dimension* $\mathrm{cld}(X)$ of X we mean the non-negative integer

$$\max\{n \in \mathbb{N} \cup \{0\} \mid X \text{ has a clique of order } n\}.$$

More generally, define $\mathcal{Q}^n X$ inductively by $\mathcal{Q}^0 X = X$ and $\mathcal{Q}^n X = \mathcal{Q}(\mathcal{Q}^{n-1}X)$ for all $n \geqslant 1$. We then define $\mathrm{cld}^n(X)$ to be $\mathrm{cld}(\mathcal{Q}^n X)$. Thus, for example, X has no triangles if and only if $\mathrm{cld}(X) \leqslant 1$ and $\mathcal{Q}X$ has no triangles if and only if $\mathrm{cld}^1(X) \leqslant 1$.

Definition 4.1. A finite graph X is *antisocial* if $\mathrm{cld}^1(X) \leqslant 1$.

The following technical lemma will later allow us, for an antisocial graph X, to pass from splittings of a subgroup of $G(X)$ to splittings of $G(X)$.

Lemma 4.2. *Let X be a finite antisocial graph. Then for all $n \geqslant 0$,*

(1) *the labelled clique graph $\mathcal{Q}X$ has a vertex v with $|v| = n + 1$ if and only if X is isomorphic in $\mathcal{X}$ to a cone $\mathcal{C}_Y Z$, where Y has an n-clique*;

(2) *the labelled clique graph $\mathcal{Q}X$ has an edge e with $|e| = n$ if and only if X is isomorphic in $\mathcal{X}$ to a proper pushout in $\mathcal{X}$ of the form $X_1 \cup_Y X_2$, such that there exists an n-clique K of Y, an m_1-clique Q_1 of X contained in X_1 $(m_1 > n)$ and an m_2-clique Q_2 of X contained in X_2 $(m_2 > n)$ with $K = Q_1 \cap Q_2$.*

Moreover, the following holds:

– *In* (1), *Y can be chosen to be either* (a) *the disjoint union of an n-clique with a single vertex or* (b) *isomorphic to $K_n \cup_{K_p} K_q$ for some integers q and p with $0 \leqslant p < q < n$.*

– *In* (2), *Y can be chosen to be a disjoint union of cliques with K and X_1 can be chosen to be a single clique of m_1 vertices.*

Proof. In both (1) and (2) it is clear from the definition of a clique graph that the second statement implies the first. It remains only to prove the converse statements.

(1) Suppose that X is a finite antisocial graph and that $\mathcal{Q}X$ has a vertex v with $|v| = n + 1$. Then $Q = \mathcal{Q}^{-1}(v)$ is an $(n + 1)$-clique of X. We write $\star(v)$ for the subgraph of $\mathcal{Q}X$ induced by v and all edges adjacent to v.

Case (i). If there is a vertex q of Q such that for all edges e of $\star(v)$ we have $q \notin l(e)$ then $X \cong \mathcal{C}_{(Q-q)}(X - q)$ which is a cone as required.

Case (ii). Otherwise, by the antisocial condition, $q \in l(e)$ for a unique edge $e = \{v, w\}$ of $\star(v)$. In this case, let X_q denote the complete subgraph of X induced by $l(w) - \{q\}$. Then $X \cong \mathcal{C}_{(Q-q)\cup X_q}(X - q)$, which is a cone as required. See Fig. 10.

(2) Suppose that X is a finite antisocial graph and that $\mathcal{Q}X$ has an edge $e = \{v_1, v_2\}$ with $|e| = n$. We may assume that X is connected (which implies that $\mathcal{Q}X$ is connected) Let $X_1 = \mathcal{Q}^{-1}(\{v_1\})$, $X_2 = \mathcal{Q}^{-1}(\mathcal{Q}X - v_1)$ and let $Y = X_1 \cap X_2$. Then X_1, X_2 and Y are as required; because of the antisocial condition we can express Y as the desired disjoint union. See Fig. 11. □

Example 4.3. Consider the graph X in Fig. 12 which is not antisocial. Its clique graph has an edge e with $|e| = 1$ but X is not a proper pushout as in (2) of the previous lemma.

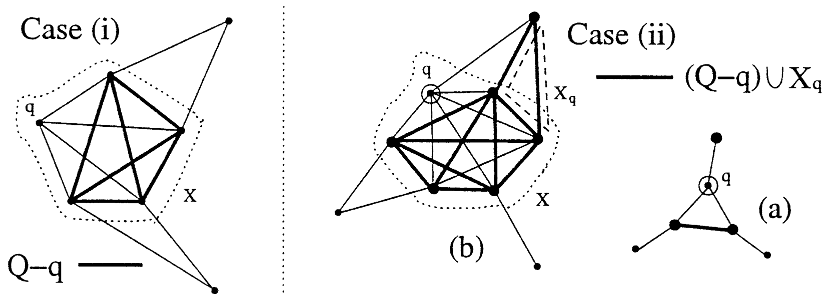


Fig. 10.

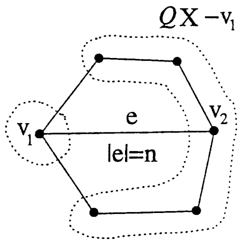


Fig. 11.

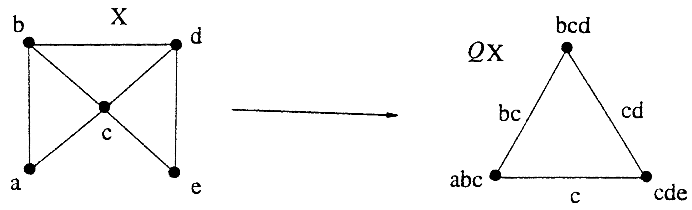


Fig. 12.

5. Betti numbers of an *X*-sphere

Since graph groups are torsion free, Stallings' ends theorem for such groups states that $e(G(X)) \geqslant 2$ if and only if $G(X)$ is freely decomposable or splits as an HNN extension over the trivial group, i.e., $G(X)$ is either freely decomposable, in which case $e(G(X)) = \infty$ or $G(X) \cong \mathbb{Z}$, in which case $e(G(X)) = 2$. Note that $G(X)$ is freely indecomposable if and only if X is connected, if and only if S^X is connected and $G(X)$ is not infinite cyclic. Thus the 0-homology of S^X reflects these splitting properties. The purpose of this paper is to investigate the significance of integral Betti number $b_n(S^X)$ in the context of splittings of $G(X)$. Note that, by Proposition 3.5, we see that for all n, $b_n(S_1(X)) = b_n S^X$. Hereafter we shall denote $b_n(S_1(X))$ by $b_n X$.

In this section we show how to calculate the integral singular homology groups $H_n S^X$ for an antisocial graph X. We first make the following observation, which follows since $\dim(\mathrm{Cube}(X)) = \mathrm{cld}(X)$.

Lemma 5.1. *If X is a finite graph and $n \geqslant \mathrm{cld}(X)$ then $b_n X = 0$.*

Assume from now on that X is antisocial. Let $X_1, \ldots, X_k$ be the connected components of X. Then $QX_1, \ldots, QX_k$ are the connected components of QX and thus $S^{X_1}, \ldots, S^{X_k}$ are those of S^X. We hence have, for all n, $b_n X = \sum_{i=1}^{k} b_n X_i$. No generality is therefore lost in assuming that X is connected. Suppose first that QX is a tree. Then we can calculate the homology of S^X inductively by using the Mayer–Vietoris sequence with only a single sphere in the intersection. The fact that the intersection is always a single sphere S^q uses the assumption that X is antisocial. Suppose that $QX = QY \cup_{\{v\}} \{e\}$ where v is a vertex and $e = \{v, w\}$ is an edge of QX. Let $p = |w| - 1$ and let $q = |e| - 1$. Then $S^X \cong S^Y \cup S^p$, $S^Y \cap S^p = S^q$ and we have the exact sequence

$$\cdots \xrightarrow{\Delta_n} H_n S^q \xrightarrow{f_n} H_n S^Y \oplus H_n S^p \xrightarrow{g_n} H_n S^X \xrightarrow{\Delta_{n-1}} \cdots.$$

So if $n \notin \{0, 1, q, q+1\}$ we have $b_n X = b_n Y + b_n S^p$, i.e., if $p \notin \{0, 1, q, q+1\}$ we have $b_p X = b_p Y + 1$ and if $n \notin \{0, 1, q, q+1, p\}$ we have $b_n X = b_n Y$.

Case $n = 0$. Since we are assuming X is connected, unless X is a single vertex, S^X is connected and so $b_0 X = 1$. If X is a single vertex then $b_0 X = 2$.

Case $n=1$. The Mayer–Vietoris sequence breaks down into split short exact sequences giving

$$H_1S^X \cong \frac{H_1S^Y \oplus H_1S^p}{\operatorname{im} f_1} \oplus \ker f_0.$$

Now we have

$$f_1 : H_1S^q \xrightarrow{i_Y^* \oplus i_p^*} H_nS^Y \oplus H_nS^p$$

where i_Y and i_p are the inclusions of S^q into S^Y and S^p. By maximality of cliques, two distinct cliques may only intersect in a maximal subgraph of strictly smaller order than either of the cliques. Thus $p>q$ and S^q is embedded in a sphere in S^Y of strictly larger dimension. This means that f_1 is always the zero map.

If $q=0$ then $f_0 : \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$ is given by $(n,m)\mapsto(n+m,n+m)$ and we have $\ker f_0 \cong \mathbb{Z}$. Thus $b_1X = b_1Y + \delta_{1,p} + 1$. Otherwise, if $q\neq 0$ then we have $f_0 : \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$ given by $n\mapsto(n,n)$ and hence $\ker f_0 \cong 0$. Therefore, in this case $b_1X = b_1Y$.

Case $n \geqslant 2$. If $q=1$ and $p=2$ then $b_2X = b_2Y + 2$. For $q\geqslant 2$, by maximality of cliques we have $\ker(f_q)\cong\mathbb{Z}$ and $\operatorname{im}(f_q)\cong 0$ and we also have $\ker(f_{q-1})\cong 0$ (since $H_{q-1}S^q\cong 0$) and $\operatorname{im}(f_{q+1})\cong 0$ (since $H_{q+1}S^q = 0$). Thus, in a similar manner to before we obtain $b_{q+1}X = b_{q+1}Y + \delta_{p,q+1} + 1$ and $b_qX = b_qY + \delta_{p,q}$.

In general, $\mathcal{Q}X$ is not a tree. Let $T=\mathcal{Q}Y$ be a maximal tree of $\mathcal{Q}X$. Let $e=\{v_1,v_2\}\in E(\mathcal{Q}X - T)$ and let $Z = \mathcal{Q}^{-1}T \cup_{\{v_1,v_2\}} \{e\}$. To calculate the homology of S^Z we take S^Y and identify two copies S_1^p and S_2^p of S^p, where $p=|e|-1$, via an equivalence relation $\sim$, i.e., $S^Z = S^Y/{\sim}$ and $S_1^p \cap S_2^p \cong S^q$ for some integer q with $-1\leqslant q\leqslant p$. Let $M(S^p)$ denote the mapping cylinder $S^p\times[0,1]$. Then $S_1^p \cong S^p\times\{0\}$, $S_2^p\cong S^p\times\{1\}$ and $S^Y\cap M(S^p) = S_1^p \coprod S_2^p$ which we shall denote by $2S^p$. We have the Mayer–Vietoris sequence

$$\cdots \xrightarrow{\Delta_n} H_n\bigl(2S^p\bigr) \xrightarrow{f_n} H_n\bigl(M\bigl(S^p\bigr)\bigr) \oplus H_nS^Y \xrightarrow{g_n} H_nS^Z \xrightarrow{\Delta_{n-1}} \cdots.$$

Now $b_0X=1$ by connectivity considerations so suppose $n\geqslant 1$. We have $b_n(2S^p) = 2\delta_{n,p}$. If $n\notin\{1,p,p+1\}$ then we have $b_nZ = b_nY$. Note that the inclusion $S_q^p \hookrightarrow S^Y$ induces the zero map on homology. When $p=0$ we have $H_1S^Z \cong H_1S^Y \oplus \ker(f_0 : \mathbb{Z}^4\to\mathbb{Z}^3)$. Here, $\ker(f_0)\cong\mathbb{Z}^2$ and we have $b_1Z = b_1Y+2$. For $p=1$ we have $\operatorname{im}(f_1)\cong\mathbb{Z}$ and hence $H_1S^Z\cong H_1S^Y\oplus\ker(f_0:\mathbb{Z}^2\to\mathbb{Z}^2)$. In this case, $\ker(f_0)\cong 0$, which gives $b_1Z=b_1Y$. For $p\geqslant 2$ we have $\operatorname{im}(f_p)\cong\mathbb{Z}$ and $\ker(f_{p-1})=0$. Thus since $H_p(M(S^p))\cong\mathbb{Z}$ we have $b_pZ=b_pY$. For $p\geqslant 1$ we also have $\ker(f_p)\cong\mathbb{Z}$ and $\operatorname{im}(f_{p+1})\cong 0$ giving $b_{p+1}Z = b_{p+1}Y+1$.

We summarise the cases when nonzero terms are added in the following formulae.

$$e\in T,\ n=p\neq 0\colon b_nX = b_nY+1, \tag{1}$$
$$e\in T,\ n=q+1\colon b_nX = b_nY+\delta_{p,q+1}+1, \tag{2}$$
$$e\notin T,\ n=p+1\colon b_nX = b_nY+\delta_{p,0}+1. \tag{3}$$

Lemma 5.2. *Let X be a finite antisocial graph and let $n>0$. If there is no edge e of $\mathcal{Q}X$ with $|e|=n$ and no vertex v of $\mathcal{Q}X$ with $|v|=n+1$ then $b_nX=0$.*

Proof. We may assume that X is connected because $b_nX = 0$ if and only if for each connected component X_i of X we have $b_nX_i = 0$. Let $\mathcal{T} = \mathcal{Q}T$ be a maximal subtree of X. We prove by induction on $|E(\mathcal{T})|$ that $b_nT = 0$. If $|E(\mathcal{T})| = 0$ then $\mathcal{Q}X$ has a single vertex v and since $n > 0$ and $n \neq |v|$ we have $b_nT = b_nS^{m_v} = 0$. If $|E(\mathcal{T})| = m > 0$ then let f be a leaf of $\mathcal{T}$ and let $\mathcal{T}' = \mathcal{T} - f = \mathcal{Q}(T')$. Then $|E(\mathcal{T}')| = m - 1$ and hence $b_n(T') = 0$ by the inductive hypothesis. We obtain b_nT from $b_n(T')$ by formulae (1) and (2), noting that $|e| = q$ and $|v| = p$. Since $n \neq q + 1$, $n \neq p$ and $n > 0$ we have $b_n(T') = b_nT = 0$.

Let $\mathcal{F} = \mathcal{Q}F$ be the subgraph of $\mathcal{Q}X$ induced by the edges not in $\mathcal{T}$. We prove by induction on $|E(\mathcal{F})|$ that $b_n(T \cup F) = 0$. If $|E(\mathcal{F})| = 0$ then $S^{T\cup F} = S^T$. Otherwise let $m' = |E(\mathcal{F})| > 0$. Let $\mathcal{F}' = \mathcal{F} - \{e\} = \mathcal{Q}(F')$ for some edge e of $\mathcal{Q}X$. Then by the inductive hypothesis $b_n(T \cup F') = 0$ and we obtain $b_n(T \cup F)$ from $b_n(T \cup F')$ by formula (3), noting that $|e| = p$. Now as $n \neq p + 1$ we have $b_n(T \cup F) = b_n(T \cup F') = 0$. Since $S^{T\cup F} = S^X$, we have shown by induction that $b_nX = 0$. $\square$

For the next lemma, it is important to understand what is meant by a *subgraph* of a graph (V, E). In the context of this paper it means a pair $(W \subset V, F \subset E)$ such that the inclusion $W \to V$ induces a graph morphism.

Lemma 5.3. *If X is a finite antisocial graph and $\mathcal{Q}Y$ is a connected subgraph of $\mathcal{Q}X$ then for all n, $b_nX \geqslant b_nY$.*

Proof. Let $X_1, \ldots, X_k$ be the connected components of X. Since $\mathcal{Q}Y$ is connected, it is a subgraph of some connected component of $\mathcal{Q}X$, say $\mathcal{Q}X_j$. Then since, for all n, $b_nX = \sum_{i=1}^{k} b_nX_i$, we have $b_nX \geqslant b_nX_j$.

Suppose that $n = 0$. Unless Y is an isolated vertex, S^Y is connected and $b_0Y = 1 = b_0X_j$. If Y is an isolated vertex then $b_0Y = 2$ and since Y is a clique of X, $X_j = Y$ and $b_0X_j = 2$. Then we have $b_0X \geqslant b_0X_j \geqslant b_0Y$.

Otherwise, assume $n \geqslant 1$. Let $\mathcal{T}$ be a maximal subtree of $\mathcal{Q}X_j/\mathcal{Q}Y$ and let $\mathcal{E} = E(\mathcal{Q}X_j/\mathcal{Q}Y) - E(\mathcal{T})$. Let $\widetilde{\mathcal{E}}$ be the lift of $\mathcal{E}$ to $E(\mathcal{Q}X_j)$ and let $\widetilde{\mathcal{T}} = \mathcal{Q}T$ be the lift of $\mathcal{T}$ to $\mathcal{Q}X_j$. We prove by induction on $|E(\mathcal{T})|$ that $b_n(Y \cup T) \geqslant b_nY$. If $|E(\mathcal{T})| = 0$ then $\mathcal{Q}Y = \mathcal{Q}X_j$ and hence $S^Y \cong S^{X_j}$. Suppose that $|E(\mathcal{T})| = m > 0$. Let v be a leaf of $\widetilde{\mathcal{T}}$ and let $\widetilde{\mathcal{T}}' = \widetilde{\mathcal{T}} - v = \mathcal{Q}(T')$. Then $|E(\widetilde{\mathcal{T}}')| = m - 1$ and by the inductive hypothesis we have $b_n(Y \cup T') \geqslant b_nY$. Now we obtain $b_n(Y \cup T)$ from $b_n(Y \cup T')$ by formulae (1) and (2), whence it is clear that $b_n(Y \cup T) \geqslant b_n(Y \cup T') \geqslant b_nY$.

Suppose that $\widetilde{\mathcal{E}}$ induces the subgraph $\overline{\mathcal{E}}$ of $\mathcal{Q}X_j$ where $\overline{\mathcal{E}} = \mathcal{Q}(\overline{E})$. We show by induction on $|\widetilde{\mathcal{E}}|$ that $b_n(Y \cup T \cup \overline{E}) \geqslant b_n(Y \cup T)$. If $|\widetilde{\mathcal{E}}| = 0$ then $S^{Y\cup T\cup \overline{E}} = S^{Y\cup T}$. If $|\widetilde{\mathcal{E}}| = m' > 0$ let $\widetilde{\mathcal{E}}' = \widetilde{\mathcal{E}} - \{e'\}$ for some edge e' of $\widetilde{\mathcal{E}}$. Let $\overline{\mathcal{E}}' = \mathcal{Q}(\overline{E}')$ be the subgraph of $\mathcal{Q}X_j$ induced by $\widetilde{\mathcal{E}}'$. Then $|\widetilde{\mathcal{E}}'| = m' - 1$ and by the inductive hypothesis $b_n(Y \cup T \cup \overline{E}') \geqslant b_n(Y \cup T)$. We obtain $b_n(Y \cup T \cup \overline{E})$ from $b_n(Y \cup T \cup \overline{E}')$ by formula (3), whence it is clear that $b_n(Y \cup T \cup \overline{E}) \geqslant b_n(Y \cup T \cup \overline{E}') \geqslant b_n(Y \cup T)$.

Thus we have $b_nX \geqslant b_n(X_j) = b_n(Y \cup T \cup \overline{E}) \geqslant b_n(Y \cup T) \geqslant b_nY$. $\square$

Example 5.4. Recall the graph X from example 4.3. The sphere S^X is as in Fig. 13.

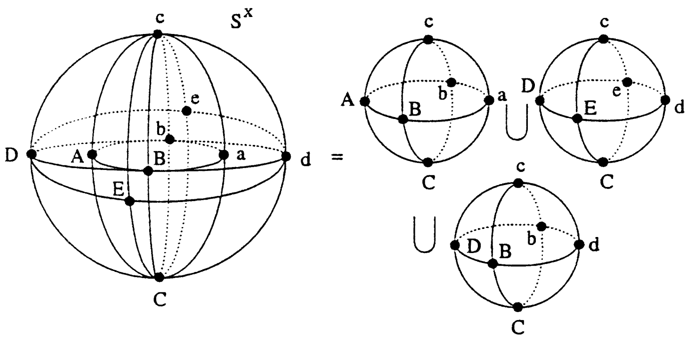


Fig. 13.

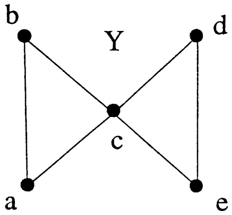


Fig. 14.

Although Lemma 4.2 failed for this example, note that nothing contradicts Lemma 5.3. A homology calculation shows that $b_1 X = 0$. This may appear to contradict Lemma 5.3 with the graph Y as in Fig. 14.

However, Y is not a subgraph in the strict sense defined above.

6. Betti numbers and subdecomposability of antisocial graph groups

Every group G can be written as $G *_H H$ if H is a subgroup of G. It is usual to exclude this triviality from the definition of a splitting: We say that a group G *splits* over a subgroup H if $G \cong_{\mathcal{G}} G_1 *_H G_2$ for subgroups G_1 and G_2 of G with $G_1 \neq H$ and $G_2 \neq H$ or if $G \cong_{\mathcal{G}} G_1 *_H$ for some subgroup G_1 of G.

In the following, a subgroup of a group G is called *maximal Abelian* if it is maximal in the lattice of Abelian subgroups of G (rather than if it is a maximal subgroup of G which is Abelian). For example, $\mathbb{Z}^2 \cong \langle a, c \rangle$ is maximal Abelian in $G = F(a, b) \times F(c, d)$ but it is not a maximal subgroup of G as it is contained in $\langle a, b, c \rangle \cong F_2 \times \mathbb{Z}$.

If $G(X)$ is a graph group then a *graphical subgroup* of $G(X)$ is a subgroup of the form $G(Y)$, where Y is a subgraph of X induced by some set of vertices.

Definition 6.1. Let X be a finite graph.

(1) We say that $G(X)$ *graphically* splits over a graphical subgroup $G(Y)$ if either
  (a) $X \cong_{\mathcal{X}} Z_1 \cup_Y Z_2$ for graphs $Z_1 \neq X$ and $Z_2 \neq X$, or
  (b) $X \cong_{\mathcal{X}} \mathcal{C}_Y Z$ for a graph Z.

(2) We say that $G(X)$ is *graphically n-subdecomposable* if it either
   (a) has a maximal Abelian graphical subgroup of rank $n+1$, or
   (b) has a graphical subgroup of the form $\mathbb{Z}^{m_1} *_{\mathbb{Z}_n} \mathbb{Z}^{m_2}$ with $m_1 > n$ and $m_2 > n$.

Theorem 6.2. *Let X be a finite antisocial graph. Then for all $n \geqslant 1$, $b_n X \neq 0$ if and only if $G(X)$ is graphically n-subdecomposable.*

Proof. Let $n \geqslant 1$. Suppose that $G(X)$ has a graphical subgroup $G(Y)$ of rank $n+1$ which is maximal Abelian. Then Y is an $(n+1)$-clique of X so $S^Y \cong S^n$ and by Lemma 5.3, $b_n X \geqslant b_n S^n = 1$. Or $G(X)$ may have a graphical subgroup of the form $H(Y) = \mathbb{Z}^{m_1} *_{\mathbb{Z}_n} \mathbb{Z}^{m_2}$ with $m_1 > n$ and $m_2 > n$. Choose these such that $\mathbb{Z}^{m_1}$ and $\mathbb{Z}^{m_2}$ are maximal Abelian. Then Y is of the form $X_1 \cup_Z X_2$ where X_1 and X_2 are cliques in X and $|Z| = n$. Now $\mathcal{Q}Y$ is a connected subgraph of $\mathcal{Q}X$ with $S^Y \cong S^{m_1-1} \cup_{S^{n-1}} S^{m_2-1}$ and for $n \geqslant 1$ we have, by (2),

$$b_n Y = b_n S^{m_1-1} + \delta_{m_2,n-1} + 1 \geqslant 1.$$

Hence, by Lemma 5.3, we have $b_n X \geqslant b_n Y \geqslant 1$.

Conversely, suppose that $b_n X \neq 0$. Then by Lemma 5.2 there is either an edge $e = \{v_1, v_2\}$ of $\mathcal{Q}X$ with $|e| = n$ or a vertex v of $\mathcal{Q}X$ with $|v| = n+1$. In the first case, let $m_1 = |v_1|$ and $m_2 = |v_2|$. Then X has as a subgraph $K_{m_1} \cup_Z K_{m_2}$ where K_{m_1} and K_{m_2} are cliques in X and $|Z| = n$. This gives rise to a subgroup $G(K_{m_1}) *_{G(Z)} G(K_{m_2})$ of X such that $G(K_{m_1})$ and $G(K_{m_2})$ are maximal Abelian and $G(Z) \cong \mathbb{Z}^n$. Thus $G(X)$ is graphically n-subdecomposable. In the second case, X has an $(n+1)$-clique Y induced by $l_{\mathcal{Q}X}(v)$ and $G(Y)$ is a graphical subgroup of rank $n+1$ which is maximal Abelian. Hence in this case $G(X)$ is also graphically n-subdecomposable.

Example 6.3. Theorem 6.2 may generalise to graphs which are not antisocial. If we consider the graph X of Example 5.4 then we see that $b_1(X') = 0$ but $G(X')$ is not graphically 1-subdecomposable (it is graphically 2-subdecomposable).

7. Decomposability of antisocial graph groups

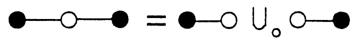
The following concept should be regarded as a generalisation of "freely decomposable or infinite cyclic" to higher dimensions in the case of graph groups.

Definition 7.1. Let X be a finite graph and let n be a nonnegative integer. We say that $G(X)$ is *graphically n-decomposable* if it splits graphically either
(1) as an HNN extension over a graphical subgroup $G(Y)$, where $G(Y)$ has a maximal Abelian graphical subgroup of rank n
(2) as a free product with amalgamation $G(X_1) *_{G(Y)} G(X_2)$ where $G(Y)$ has a maximal Abelian subgroup of rank n but $G(Y)$ is maximal Abelian in neither $G(X_1)$ nor $G(X_2)$.

Suppose that X is a finite graph. Then X is graphically 0-decomposable if and only if either X is freely decomposable or X is isomorphic to $\mathbb{Z}$ (which is an HNN extension over the trivial group). To see this, suppose that X is graphically 0-decomposable. This may be because $G(X)$ is an HNN extension over the trivial group (the only graph with a 0-clique is the empty graph), which means that X has a connected component with a single vertex. Hence $G(X)$ is either isomorphic to $\mathbb{Z}$ or to a nontrivial free product of a graph group with $\mathbb{Z}$. On the other hand it may be because $G(X)$ decomposes graphically as a free product with amalgamation $A *_C B$ over a graphical subgraph $C = G(Y)$, where Y has a clique of order 0, hence is the trivial group, and the condition that $G(Y)$ is maximal Abelian in neither factor tells us that both factors are nontrivial. Thus $G(X)$ is freely decomposable. Conversely, if $G \cong \mathbb{Z}$ then H is an HNN extension over the trivial group; if G is freely decomposable then G is a free product with amalgamation as required.

Example 7.2. F_3 splits graphically over $\mathbb{Z}$ as

but this is not a splitting as in the definition of graphical 1-decomposability since the group $Y \cong \mathbb{Z}$ which we are splitting over is maximal Abelian in one of the factors (in fact, in both of them). On the other hand, $F_2 \times \mathbb{Z}$ has a $\mathbb{Z}$-splitting which satisfies the definition of graphical 1-decomposability:



so this is graphically 1-decomposable. Note that in this case if Y is the graph above then we have $b_1 Y \neq 0$.

A higher dimensional example is given by the graph groups on the graphs X_1 and X_2. See Fig. 15.

Both $G(X_1)$ and $G(X_2)$ split graphically over $\mathbb{Z}^2$ as free products with amalgamation. See Fig. 16.

The group $G(X_1)$ is not graphically 2-decomposable, whereas $G(X_2)$ is. Note that $b_2(X_1) = 0$ as $\dim(\mathrm{Cube}(X_1)) = 2$ whereas $b_2(X_2) = 3$.

In actual fact, however, the graphically 1-decomposable and 2-decomposable examples above are decomposable for two reasons, as they also split as appropriate HNN extensions. The graphs X_3 and X_4 in Fig. 17 satisfy the properties that X_3 is graphically 1-decomposable without being a graphical HNN extension over $\mathbb{Z}$ and X_4 is graphically 2-decomposable without being a graphical HNN extension over $\mathbb{Z}^2$.

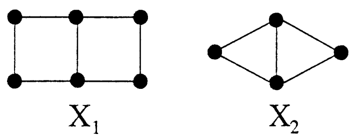


Fig. 15.

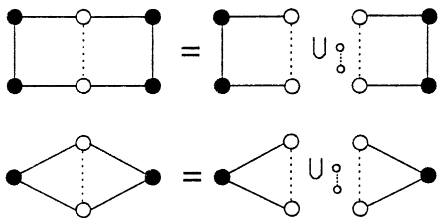


Fig. 16.

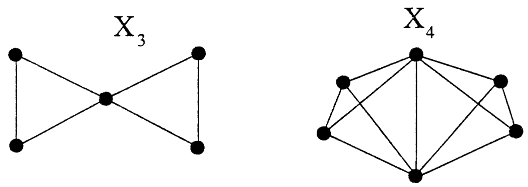


Fig. 17.

Suppose that $G(X)$ has either 2 or infinitely many ends, i.e., $b_0X > 1$. Since graph groups are torsion free, Stallings' ends theorem says that such groups are HNN extensions over the trivial group or freely indecomposable, i.e., they are (graphically) 0-decomposable. The correct analogue of this in higher dimensions is that $b_nX \neq 0$:

Theorem 7.3. *Let X be a finite antisocial graph and let $n \geqslant 1$ be an integer. Then the following are equivalent.*

(1) *The Betti number $b_nX \neq 0$.*
(2) *The group $G(X)$ is graphically n-subdecomposable.*
(3) *The group $G(X)$ is graphically n-decomposable.*
(4) *The group $G(X)$ splits graphically as one of the following*
   (a) *A free product $\mathbb{Z}^{m_1} *_K H$ with amalgamation where*
      – *K is a free product of finitely many free Abelian groups of finite rank with a free Abelian group A of rank n,*
      – *H is a graph group*
      – *A is not maximal Abelian in H and $m_1 > n$.*
   (b) *An HNN extension over $\mathbb{Z}^n *_{\mathbb{Z}^p} \mathbb{Z}^q$, where $0 \leqslant p < q < n$.*

Proof. The equivalence of (1) and (2) is Theorem 6.2. The equivalence of (2), (3) and (4) results from translating Lemma 4.2 into the context of graph groups. □

For graphs which are not antisocial, it may still be the case that (1) and some version of (3) are equivalent, but a different method of proof from using the Mayer–Vietoris sequence may be necessary. (4) is a refined statement I would only expect to be true for antisocial graphs.

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